J. Math. Anal. Appl. 386 (2012) 1–9

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Inverse Sturm–Liouville problems with finite spectrum

Qingkai Kong *, Anton Zettl

Department of Mathematics, Northern Illinois University, DeKalb, IL 60115, USA

ARTICLE INFO

Article history:
Received 13 November 2009
Available online 5 July 2011
Submitted by S. Fulling

Keywords:
Inverse Sturm–Liouville problems
Inverse matrix eigenvalue problems
Finite spectrum

ABSTRACT

We study inverse Sturm–Liouville problems of Atkinson type whose spectrum consists entirely of a finite set of eigenvalues. We show that given two finite sets of interlacing real numbers there exists a class of Sturm–Liouville equations of Atkinson type such that the two sets of numbers are the eigenvalues of their associated Sturm–Liouville problems with two different separated boundary conditions. Parallel results are also obtained for real coupled boundary conditions. Our approach is to use the equivalence between Sturm–Liouville problems of Atkinson type and matrix eigenvalue problems and to apply our development of the well-known theory for inverse matrix eigenvalue problems.

1. Introduction

In [7] Kong, Wu, and Zettl constructed a class of regular self-adjoint Sturm–Liouville problems (SLPs) whose spectrum consists entirely of a finite number of eigenvalues. Since such problems were initiated by Atkinson [1], following Kong, Volkmer, and Zettl [6], we refer to them as problems of Atkinson type. In [6], the authors systematically explored the relationship between regular self-adjoint SLPs of Atkinson type and matrix eigenvalue problems in the form

$$DX = \lambda WX, \tag{1.1}$$

where D and W are $n \times n$ matrices over the reals $\mathbb{R}$ and W is diagonal.

They considered the problem consisting of the Sturm–Liouville equation

$$-(py')' + qy = \lambda wy \quad \text{on } (a,b), \tag{1.2}$$

where $-\infty < a < b < \infty$ and $r := \frac{1}{p}$, $q, w \in L(a,b)$, the set of real integrable functions on (a,b), and the regular self-adjoint boundary condition (BC)

$$AY(a) + BY(b) = 0, \quad Y = [y, py']^T,$$

where A, B are 2×2 complex matrices satisfying

$$\operatorname{rank}(A,B) = 2, \qquad AEA^* = BEB^* \quad \text{with } E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.3}$$

It is well known [12, p. 71] that such BCs (1.3) fall into two disjoint classes: separated and coupled. The separated ones have the canonical representation

* Corresponding author.
E-mail addresses: kong@math.niu.edu (Q. Kong), zettl@math.niu.edu (A. Zettl).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.083

$$\begin{aligned}&\cos\alpha\, y(a) - \sin\alpha\big(py'\big)(a) = 0, \quad 0 \leqslant \alpha < \pi,\\ &\cos\beta\, y(b) - \sin\beta\big(py'\big)(b) = 0, \quad 0 < \beta \leqslant \pi;\end{aligned} \tag{1.4}$$

and the real coupled conditions have the canonical representation

$$Y(b) = K\,Y(a) \quad \text{with } K = (k_{ij}) \in SL_2(\mathbb{R}), \tag{1.5}$$

i.e., $k_{ij} \in \mathbb{R}$, $1 \leqslant i, j \leqslant 2$ and $\det(K) = 1$. The Sturm–Liouville equation (1.2) is said to be of Atkinson type if for some positive integer $n > 1$, there exists a partition of the interval (a, b)

$$a = a_0 < b_0 < a_1 < b_1 < \cdots < a_n < b_n = b \tag{1.6}$$

such that

$$r = 0 \quad \text{on } [a_i, b_i],\ i = 0, \ldots, n, \qquad \int_{b_{i-1}}^{a_i} r > 0, \quad i = 1, 2, \ldots, n; \tag{1.7}$$

$$q = 0 \quad \text{on } [b_{i-1}, a_i],\ i = 1, \ldots, n; \tag{1.8}$$

and

$$w = 0 \quad \text{on } [b_{i-1}, a_i],\ i = 1, \ldots, n, \qquad \int_{a_i}^{b_i} w > 0, \quad i = 0, 1, \ldots n. \tag{1.9}$$

SLP (1.2), (1.3) is said to be of Atkinson type if Eq. (1.2) is of Atkinson type and BC (1.3) is self-adjoint.

It was established in [6] that any SLP of Atkinson type with either separated or real coupled BC has an equivalent matrix eigenvalue problem of the form of (1.1) in the sense that the two problems have exactly the same eigenvalues. As a result, in sharp contrast to the classical regular SLPs, the spectrum of a SLP of Atkinson type consists entirely of a finite number of eigenvalues. In fact, the number of eigenvalues of SLP (1.2), (1.3) is either $n + 1$, n, or $n - 1$ with n given in (1.6), depending on the BC. Moreover, all eigenvalues are real even for the case when r and w change sign, as long as (1.7)–(1.9) hold.

In this paper, we will investigate the inverse spectral theory for SLP of Atkinson type. Our main approach is to use the equivalence between SLP (1.2), (1.3) and the matrix eigenvalue problem (1.1) established in [6] and to apply our development of the well-known theory, see Ferguson [4], Hochstadt [5], and Xu [11], for inverse matrix eigenvalue problems. Since we have not found such development in the literature, we will present the detailed results and proofs.

This paper is organized as follows: Our main results are stated in Section 2, and proofs are given in Section 4 after our extensions of the well-known inverse matrix eigenvalue problems are developed in Section 3.

2. Main results

To present our results, we use the following lemma which highlights the fact that every SLP of Atkinson type is equivalent to a SLP with piecewise constant coefficients, see Theorem 2.4 in [6].

Lemma 2.1. *Assume $r, p, q \in L(a, b)$ satisfy (1.7)–(1.9). Let*

$$p_i = \left(\int_{b_{i-1}}^{a_i} r\right)^{-1}, \quad i = 1, 2, \ldots n; \qquad q_i = \int_{a_i}^{b_i} q, \qquad w_i = \int_{a_i}^{b_i} w, \quad i = 0, 1, \ldots n. \tag{2.1}$$

Define piecewise constant functions $\bar{p}, \bar{q}$, and $\bar{w}$ on (a, b) by

$$\bar{p}(t) = \begin{cases} p_k(a_i - b_{i-1}),\ t \in [b_{i-1}, a_i],\ i = 1, 2, \ldots, n,\\ \infty,\ t \in [a_i, b_i],\ i = 0, 1, \ldots, n;\end{cases}$$

$$\bar{q}(t) = \begin{cases} q_i/(b_i - a_i),\ t \in [a_i, b_i],\ i = 0, 1, \ldots, n,\\ 0,\ t \in [b_{i-1}, a_i],\ i = 1, 2, \ldots, n;\end{cases}$$

$$\bar{w}(t) = \begin{cases} w_i/(b_i - a_i),\ t \in [a_i, b_i],\ i = 0, 1, \ldots, n,\\ 0,\ t \in [b_{i-1}, a_i],\ i = 1, 2, \ldots, n.\end{cases}$$

Here $\bar{p}(t) = \infty$ on $[a_i, b_i]$ means that $r(t) = 0$ on $[a_i, b_i]$. Suppose the self-adjoint BC (1.3) is either separated or real coupled. Then SLP (1.2), (1.3) has exactly the same eigenvalues as the SLP consisting of the equation with piecewise constant coefficients

$$-\big(\bar{p}y'\big)' + \bar{q}y = \lambda\bar{w}y \quad \text{on } (a, b) \tag{2.2}$$

and the same BC (1.3).

By Lemma 2.1 we see that for a fixed BC (1.3) and a given partition of the interval (a, b), there is a family of SLPs of Atkinson type which have exactly the same eigenvalues as SLP (2.2), (1.3). Such a family is called the *equivalent family* of SLPs (2.2), (1.3). It is clear that for an equivalent family of SLPs (2.2), (1.3), the p_k, q_k, and w_k defined by (2.1) are all the same.

For a given Eq. (1.2) with coefficients satisfying (1.7)–(1.9), $\sigma(\alpha, \beta)$ denotes the spectrum of SLP consisting of Eq. (1.2) and the separated BC (1.4) and $\sigma(K)$ the spectrum of SLP consisting of Eq. (1.2) and the real coupled BC (1.5).

In the sequel we always assume $k \in \mathbb{N}$ such that $k > 2$. We now state our results on the inverse problem of SLP (1.2), (1.3). The first two theorems are for the separated BCs in the form of (1.4).

Theorem 2.1. *Let $\alpha, \beta \in (0, \pi)$, and let $\{\lambda_i\colon i = 1, \ldots, k\}$ and $\{\mu_i\colon i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying the strict interlacing relation*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{k-1} < \mu_{k-1} < \lambda_k. \tag{2.3}$$

Let $n = k - 1$. Then for any $-\infty < a < b < \infty$, any partition (1.6), *and any $w \in L(a, b)$ satisfying* (1.9), *we have the following*:

(a) *There exist $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.4) *has the spectrum*

$$\sigma(\alpha, \beta) = \{\lambda_i\colon i = 1, \ldots, k\} \quad \text{and} \quad \sigma(0, \beta) = \{\mu_i\colon i = 1, \ldots, k-1\}.$$

(b) *There exist $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.4) *has the spectrum*

$$\sigma(\alpha, \beta) = \{\lambda_i\colon i = 1, \ldots, k\} \quad \text{and} \quad \sigma(\alpha, \pi) = \{\mu_i\colon i = 1, \ldots, k-1\}.$$

Furthermore, the equivalent families in (a) *and* (b) *are uniquely determined.*

Theorem 2.2. *Let $\alpha, \beta \in (0, \pi)$, and let $\{\lambda_i\colon i = 1, \ldots, k\}$ and $\{\mu_i\colon i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying the strict interlacing relation* (2.3). *Let $n = k$. Then for any $-\infty < a < b < \infty$, any partition* (1.6), *and any $w \in L(a, b)$ satisfying* (1.9), *we have the following*:

(a) *There exist $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.4) *has the spectrum*

$$\sigma(0, \beta) = \{\lambda_i\colon i = 1, \ldots, k\} \quad \text{and} \quad \sigma(0, \pi) = \{\mu_i\colon i = 1, \ldots, k-1\}.$$

(b) *There exist $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.4) *has the spectrum*

$$\sigma(\alpha, \pi) = \{\lambda_i\colon i = 1, \ldots, k\} \quad \text{and} \quad \sigma(0, \pi) = \{\mu_i\colon i = 1, \ldots, k-1\}.$$

Furthermore, the equivalent families in (a) *and* (b) *are uniquely determined.*

Corollary 2.1. *Let $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$, and let $\{\lambda_i\colon i = 1, \ldots, k\}$ be real numbers satisfying*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k.$$

Then there is an infinite number of equivalent families of SLPs (1.2), (1.4) *which has the spectrum*

$$\sigma(\alpha, \pi) = \{\lambda_i\colon i = 1, \ldots, k\}.$$

Proof. This follows immediately from Theorems 2.1 and 2.2. □

Corollary 2.1 extends a result in [7] which shows that given any k distinct real numbers, there is a SLP whose spectrum is arbitrarily close to the given numbers.

The next two theorems are for the real coupled BCs in the form of (1.5).

Theorem 2.3. *Let $\{\lambda_i\colon i = 1, \ldots, k\}$ and $\{\mu_i\colon i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying the three following conditions*:

(i) $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \lambda_{k-1} \leqslant \mu_{k-1} \leqslant \lambda_k$,
(ii) $\mu_i \neq \mu_j$ *if* $i \neq j$,
(iii) *there exists a $d > 0$ such that for all $j = 1, 2, \ldots, k-1$*

$$\prod_{i=1}^{k} |\mu_j - \lambda_i| \geqslant 2d\bigl[1 + (-1)^{k+1-j}\bigr]. \tag{2.4}$$

Let $n = k - 1$. Then for any $-\infty < a < b < \infty$, any partition (1.6)*, and any $w \in L(a, b)$ satisfying* (1.9)*, we have the following*:

(a) *For any $\beta \in (0, \pi)$, there exist $K = (k_{ij}) \in SL_2(\mathbb{R})$ satisfying $k_{12} < 0$ and $\cot\beta = k_{22}/k_{12}$ and $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.5) *has the spectrum*

$$\sigma(K) = \{\lambda_i\colon\ i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(0, \beta) = \{\mu_i\colon\ i = 1, \ldots, k-1\}.$$

(b) *For any $\alpha \in (0, \pi)$, there exist $K = (k_{ij}) \in SL_2(\mathbb{R})$ satisfying $k_{12} < 0$ and $\cot\alpha = -k_{11}/k_{12}$ and $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.5) *has the spectrum*

$$\sigma(K) = \{\lambda_i\colon\ i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(\alpha, \pi) = \{\mu_i\colon\ i = 1, \ldots, k-1\}.$$

Theorem 2.4. *Let $K = (k_{ij}) \in SL_2(\mathbb{R})$ with $k_{12} = 0$ and $k_{11} > 0$, and let $\{\lambda_i\colon\ i = 1, \ldots, k\}$ and $\{\mu_i\colon\ i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying the conditions* (i)–(iii) *in Theorem* 2.3*. Let $n = k$. Then for any $-\infty < a < b < \infty$ and $w \in L(a, b)$ satisfying* (1.9)*, there exist $r, q \in L(a, b)$ satisfying* (1.7) *and* (1.8) *such that the associated equivalent family of SLPs* (1.2), (1.5) *has the spectrum*

$$\sigma(K) = \{\lambda_i\colon\ i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(0, \pi) = \{\mu_i\colon\ i = 1, \ldots, k-1\}.$$

Remark 2.1. Conditions (i)–(iii) in Theorem 2.3 imply that

(a) the multiplicities of the numbers in $\{\lambda_i\colon\ i = 1, \ldots, k\}$ are at most 2,
(b) all numbers in $\{\mu_i\colon\ i = 1, \ldots, k-1\}$ are distinct,
(c) for all $j = 1, \ldots, k-1$, if $\mu_j = \lambda_i$ for some $i \in \{1, \ldots, k\}$, then j must be even when k is even, and j must be odd when k is odd.

Remark 2.2. In the conclusions of Theorems 2.3 and 2.4, the existence of inverse problems is guaranteed for an arbitrary BC matrix K with $k_{12} = 0$ and $k_{11} > 0$, but only for some BC matrix K with $k_{12} \neq 0$. The case when $k_{12} = 0$ and $k_{11} < 0$ remains unsolved. In particular, the semi-periodic case $k_{11} = -1 = k_{22}$ and $k_{12} = 0 = k_{21}$ remains open.

Remark 2.3. In all four Theorems 2.1–2.4, $\{\mu_i\colon\ i = 1, \ldots, k-1\}$ are eigenvalues for a Dirichlet BC either at a or b. This shows that Dirichlet BCs play a special role in the inverse spectral theory of SLPs of Atkinson type.

3. Inverse matrix eigenvalue problems

In this section, we discuss the inverse matrix eigenvalue problems for Jacobi and cyclic Jacobi matrices. Our results are extensions of those from Xu [11, Chapter 2] to more general forms of inverse matrix eigenvalue problems to meet the need for the proofs of the main results in Section 2.

Let $\mathbb{M}_k$ be the set of $k \times k$ matrices over the reals. For any $C \in \mathbb{M}_k$, we denote by $\sigma(C)$ the set of eigenvalues of C. Furthermore, let C_1 be the principal submatrix obtained from C by removing its first row and column, and C^1 its submatrix obtained from C by removing the k-th row and column.

For any $C, D \in \mathbb{M}_k$, we say that λ^* is an eigenvalue of the matrix-pair (C, D) if there exists a nontrivial vector $u \in \mathbb{R}^k$ such that $(C - \lambda^* D)u = 0$. We denote by $\sigma(C, D)$ the set of eigenvalues of (C, D). Clearly, $\lambda^* \in \sigma(C)$ if and only if $\lambda^* \in \sigma(C, I_k)$, where I_k is the identity matrix in $\mathbb{M}_k$.

We first consider symmetric matrices in $\mathbb{M}_k$ of the form

$$\begin{bmatrix} c_1 & d_1 & & & & \\ d_1 & c_2 & d_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & d_{k-2} & c_{k-1} & d_{k-1} \\ & & & & d_{k-1} & c_k \end{bmatrix}. \tag{3.1}$$

Definition 3.1. A matrix $J \in \mathbb{M}_k$ in the form of (3.1) is called a positive Jacobi matrix if $d_i > 0$ for all $i = 1, 2, \ldots, k$; and it is called a negative Jacobi matrix if $d_i < 0$ for all $i = 1, 2, \ldots, k$. We say that J is a Jacobi matrix if it is either a positive or a negative Jacobi matrix.

Now we state a lemma from Xu [11, Theorem 2.3.3] on the inverse eigenvalue problem for positive Jacobi matrices.

Lemma 3.1. *Let $\{\lambda_i\colon\ i = 1, \ldots, k\}$ and $\{\mu_i\colon\ i = 1, \ldots, k-1\}$ be two sets of real numbers satisfying the strict interlacing relation* (2.3)*. Then there exists a unique positive Jacobi matrix $J \in \mathbb{M}_k$ such that*

$$\sigma(J) = \{\lambda_i\colon\ i = 1, \ldots, k\} \quad \textit{and} \quad \sigma(J_1) = \{\mu_i\colon\ i = 1, \ldots, k-1\}.$$

The next two theorems are extensions of Lemma 3.1.

Theorem 3.1. *Let $\{\lambda_i\colon i=1,\ldots,k\}$ and $\{\mu_i\colon i=1,\ldots,k-1\}$ be two sets of real numbers satisfying the strict interlacing relation* (2.3)*. Let $W=\operatorname{diag}(w_1,\ldots,w_k)$ be a diagonal matrix with $w_i>0$ for $i=1,\ldots,k$. Then there exists a unique positive Jacobi matrix $M\in\mathbb{M}_k$ such that*

$$\sigma(M,W)=\{\lambda_i\colon i=1,\ldots,k\}\quad\text{and}\quad\sigma(M_1,W_1)=\{\mu_i\colon i=1,\ldots,k-1\}. \tag{3.2}$$

Proof. By Lemma 3.1 there exists a unique positive Jacobi matrix $J\in\mathbb{M}_k$ such that

$$\sigma(J)=\{\lambda_i\colon i=1,\ldots,k\}\quad\text{and}\quad\sigma(J_1)=\{\mu_i\colon i=1,\ldots,k-1\}.$$

Hence for each $\lambda=\lambda_i$, $i=1,\ldots,k$, there exists a nontrivial $u\in\mathbb{R}^k$ such that $(J-\lambda I_k)u=0$. Let $R=\sqrt{W}:=\operatorname{diag}(\sqrt{w_1},\ldots,\sqrt{w_k})$ and let $u=R\tilde{u}$. Multiplying the above equation by R we get

$$\big(RJR-\lambda R^2\big)\tilde{u}=0,\quad\text{i.e.,}\quad(M-\lambda W)\tilde{u}=0,$$

where $M=RJR$. Clearly, $\lambda\in\sigma(M,W)$ and M is also a positive Jacobi matrix. Similarly, for each $\mu=\mu_i$, $i=1,\ldots,k-1$, there exists a nontrivial $v\in\mathbb{R}^{k-1}$ such that $(J_1-\lambda I_{k-1})v=0$. We let $v=R_1\tilde{v}$. Then multiplying the above equation by R_1 we obtain $(R_1J_1R_1-\mu R_1^2)\tilde{v}=0$. We note that $M_1=R_1J_1R_1$ and $W_1=R_1^2$. This shows that $\mu\in\sigma(M_1,W_1)$. Thus,

$$\sigma(J)\subset\sigma(M,W)\quad\text{and}\quad\sigma(J_1)\subset\sigma(M_1,W_1).$$

By reversing the steps in this argument we see that

$$\sigma(J)\supset\sigma(M,W)\quad\text{and}\quad\sigma(J_1)\supset\sigma(M_1,W_1).$$

Therefore,

$$\sigma(J)=\sigma(M,W)\quad\text{and}\quad\sigma(J_1)=\sigma(M_1,W_1).$$

To show the uniqueness, let M be any positive Jacobi matrix satisfying (3.2). Then

$$\sigma\big(R^{-1}MR^{-1}\big)=\{\lambda_i\colon i=1,\ldots,k\}\quad\text{and}\quad\sigma\big(R_1^{-1}M_1R_1^{-1}\big)=\{\mu_i\colon i=1,\ldots,k-1\}.$$

Note that $R^{-1}MR^{-1}$ is a positive Jacobi matrix and $(R^{-1}MR^{-1})_1=R_1^{-1}M_1R_1^{-1}$. By Lemma 3.1, $R^{-1}MR^{-1}$, and hence M, is uniquely determined. This completes the proof. □

Theorem 3.2. *Let $\{\lambda_i\colon i=1,\ldots,k\}$ and $\{\mu_i\colon i=1,\ldots,k-1\}$ be two sets of real numbers satisfying the strict interlacing property* (2.3)*. Let $W=\operatorname{diag}(w_1,\ldots,w_k)$ be a diagonal matrix with $w_i>0$ for $i=1,\ldots,k$. Then there exists a unique negative Jacobi matrix $M\in\mathbb{M}_k$ such that*

$$\sigma(M,W)=\{\lambda_i\colon i=1,\ldots,k\}\quad\textit{and}\quad\sigma(M_1,W_1)=\{\mu_i\colon i=1,\ldots,k-1\}.$$

Proof. Let $\xi_i=-\lambda_{k+1-i}$, $i=1,\ldots,k+1$ and $\nu_i=-\mu_{k-i}$, $i=1,\ldots,k-1$. Then

$$\xi_1<\eta_1<\xi_2<\eta_2<\cdots<\xi_{k-1}<\eta_{k-1}<\xi_k.$$

By Theorem 3.1, there exists a unique positive Jacobi matrix $M\in\mathbb{M}_k$ such that

$$\sigma(M,W)=\{\xi_i\colon i=1,\ldots,k\}\quad\text{and}\quad\sigma(M_1,W_1)=\{\eta_i\colon i=1,\ldots,k-1\}.$$

It follows that

$$\sigma(-M,W)=\{-\xi_i\colon i=1,\ldots,k\}=\{\lambda_i\colon i=1,\ldots,k\}$$

and

$$\sigma(-M_1,W_1)=\{-\eta_i\colon i=1,\ldots,k-1\}=\{\mu_i\colon i=1,\ldots,k-1\}.$$

Note that M is a negative Jacobi matrix and $(-M)_1=-M_1$. The proof is complete. □

Corollary 3.1. *Theorems* 3.1 *and* 3.2 *hold when M_1 and W_1 are replaced by M^1 and W^1, respectively.*

Proof. Let $\widetilde{M}=GMG$ and $\widetilde{W}=GWG$ with $G=\begin{bmatrix} & & & 1\\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{bmatrix}$. Then the i-th row of $\widetilde{M}$ is the same as the $(k-i)$-th row of M, and the same for the columns. Hence $\widetilde{M}_1=M^1$. Similarly, $\widetilde{W}_1=W^1$. Therefore, the conclusion follows from Theorems 3.1 and 3.2 with M and W replaced by $\widetilde{M}$ and $\widetilde{W}$, respectively. □

Corollary 3.2. *Theorems* 3.1 *and* 3.2 *hold when $w_i>0$ is replaced by $w_i<0$ for $i=1,\ldots,k$.*

Proof. Let $\{\lambda_i\colon i=1,\dots,k\}$ and $\{\mu_i\colon i=1,\dots,k-1\}$ be two sets of real numbers satisfying the strict interlacing relation (2.3). By Theorem 3.1, there exists a unique positive Jacobi matrix $M \in \mathbb{M}_k$ such that

$$\sigma(M,-W)=\{-\lambda_i\colon i=1,\dots,k\} \quad \text{and} \quad \sigma(M_1,W_1)=\{-\mu_i\colon i=1,\dots,k-1\}.$$

Hence

$$\sigma(M,W)=\{\lambda_i\colon i=1,\dots,k\} \quad \text{and} \quad \sigma(M_1,W_1)=\{\mu_i\colon i=1,\dots,k-1\}.$$

This shows that Theorem 3.1 holds when $w_i>0$ is replaced by $w_i<0$ for $i=1,\dots,k$. The same argument applies to Theorem 3.2. $\square$

We next consider symmetric matrices in $\mathbb{M}_k$ of the form

$$\begin{bmatrix} c_1 & d_1 & & & & d_k \\ d_1 & c_2 & d_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & d_{k-2} & c_{k-1} & d_{k-1} \\ d_k & & & & d_{k-1} & c_k \end{bmatrix}. \tag{3.3}$$

Definition 3.2. A matrix $J \in \mathbb{M}_k$ in the form of (3.3) is called a positive cyclic Jacobi matrix if $d_i>0$ for all $i=1,2,\dots,k-1$; and it is called a negative cyclic Jacobi matrix if $d_i<0$ for all $i=1,2,\dots,k-1$. We say that J is a cyclic Jacobi matrix if it is either a positive or a negative cyclic Jacobi matrix. Cyclic Jacobi matrices are also called periodic Jacobi matrices.

Now we state another lemma from Xu [11, Theorem 2.8.3] on the inverse eigenvalue problem for positive cyclic Jacobi matrices. Note that the uniqueness is not guaranteed by this lemma, see [11, p. 78].

Lemma 3.2. *Let $\{\lambda_i\colon i=1,\dots,k\}$ and $\{\mu_i\colon i=1,\dots,k-1\}$ be two sets of real numbers satisfying conditions* (i)–(iii) *of Theorem* 2.3. *Then for any $d>0$ satisfying* (2.4)*, there exists a positive cyclic Jacobi matrix J in the form of* (3.3) *such that $\prod_{i=1}^k d_i = d$ and*

$$\sigma(J)=\{\lambda_i\colon i=1,\dots,k\} \quad \textit{and} \quad \sigma(J_1)=\{\mu_i\colon i=1,\dots,k-1\}.$$

The theorems below are extensions of Lemma 3.2. Since the proofs are similar to those of Theorems 3.1 and 3.2, we omit the details.

Theorem 3.3. *Let $\{\lambda_i\colon i=1,\dots,k\}$ and $\{\mu_i\colon i=1,\dots,k-1\}$ be two sets of real numbers satisfying conditions* (i)–(iii) *of Theorem* 2.3. *Let $W=\operatorname{diag}(w_1,\dots,w_k)$ be a diagonal matrix with $w_i>0$ for $i=1,\dots,k$. Then for any $d>0$ satisfying* (2.4)*, there exists a positive cyclic Jacobi matrix N in the form of* (3.3) *such that $\prod_{i=1}^k d_i = d$ and*

$$\sigma(N,W)=\{\lambda_i\colon i=1,\dots,k\} \quad \textit{and} \quad \sigma(N_1,W_1)=\{\mu_i\colon i=1,\dots,k-1\}.$$

Theorem 3.4. *Let $\{\lambda_i\colon i=1,\dots,k\}$ and $\{\mu_i\colon i=1,\dots,k-1\}$ be two sets of real numbers satisfying conditions* (i)–(iii) *of Theorem* 2.3. *Let $W=\operatorname{diag}(w_1,\dots,w_k)$ be a diagonal matrix with $w_i>0$ for $i=1,\dots,k$. Then for any $d>0$ satisfying* (2.4)*, there exists a negative cyclic Jacobi matrix N in the form of* (3.3) *such that $\prod_{i=1}^k d_i = d$ and*

$$\sigma(N,W)=\{\lambda_i\colon i=1,\dots,k\} \quad \textit{and} \quad \sigma(N_1,W_1)=\{\mu_i\colon i=1,\dots,k-1\}.$$

With the same arguments as in Corollaries 3.1 and 3.2, we have

Corollary 3.3. (a) *The conclusions of Theorems* 3.3 *and* 3.4 *hold when M_1 and W_1 are replaced by M^1 and W^1, respectively.*
(b) *The conclusions of Theorems* 3.3 *and* 3.4 *hold when $w_i>0$ is replaced by $w_i<0$ for $i=1,\dots,k$.*

4. Proofs of the main results

To prove Theorems 2.1 and 2.2 we use, in addition to Theorems 3.1 and 3.2 for the inverse Jacobi matrix problems, the following results from [6, Corollaries 2.1–2.4] on the equivalence between Sturm–Liouville problems of Atkinson type and matrix eigenvalue problems.

Lemma 4.1. *Consider the separated BC* (1.4) *with $\alpha,\beta \in (0,\pi)$. Define an $(n+1)\times(n+1)$ Jacobi matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1+\cot\alpha & -p_1 & & & & \\ -p_1 & p_1+p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1}+p_n & -p_n \\ & & & & -p_n & p_n-\cot\beta \end{bmatrix} \tag{4.1}$$

and diagonal matrices

$$Q_{\alpha\beta} = \operatorname{diag}(q_0, q_1, \ldots, q_{n-1}, q_n), \qquad W_{\alpha\beta} = \operatorname{diag}(w_0, w_1, \ldots, w_{n-1}, w_n). \tag{4.2}$$

Then the spectrum $\sigma(\alpha, \beta)$ of SLP (2.2), (1.4) *and the spectrum $\sigma(P_{\alpha\beta} + Q_{\alpha\beta}, W_{\alpha\beta})$ of the matrix-pair $(P_{\alpha\beta} + Q_{\alpha\beta}, W_{\alpha\beta})$ are the same.*

Lemma 4.2. *Consider the separated BC* (1.4) *with $\alpha = 0$ and $\beta \in (0, \pi)$. Define an $n \times n$ Jacobi matrix*

$$P_{0\beta} = \begin{bmatrix} p_1 + p_2 & -p_2 & & & \\ -p_2 & p_2 + p_3 & -p_3 & & \\ & \cdots & \cdots & \cdots & \\ & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ & & & -p_n & p_n - \cot\beta \end{bmatrix} \tag{4.3}$$

and diagonal matrices

$$Q_{0\beta} = \operatorname{diag}(q_1, \ldots, q_{n-1}, q_n), \qquad W_{0\beta} = \operatorname{diag}(w_1, \ldots, w_{n-1}, w_n). \tag{4.4}$$

Then the spectrum $\sigma(0, \beta)$ of SLP (2.2), (1.4) *and the spectrum $\sigma(P_{0\beta} + Q_{0\beta}, W_{0\beta})$ of the matrix-pair $(P_{0\beta} + Q_{0\beta}, W_{0\beta})$ are the same.*

Lemma 4.3. *Consider the separated BC* (1.4) *with $\alpha \in (0, \pi)$ and $\beta = \pi$. Define an $n \times n$ Jacobi matrix*

$$P_{\alpha\pi} = \begin{bmatrix} p_1 + \cot\alpha & -p_1 & & & \\ -p_1 & p_1 + p_2 & -p_2 & & \\ & \cdots & \cdots & \cdots & \\ & & -p_{n-2} & p_{n-2} + p_{n-1} & -p_{n-1} \\ & & & -p_{n-1} & p_{n-1} + p_n \end{bmatrix}$$

and diagonal matrices

$$Q_{\alpha\pi} = \operatorname{diag}(q_0, q_1, \ldots, q_{n-1}), \qquad W_{\alpha\pi} = \operatorname{diag}(w_0, w_1, \ldots, w_{n-1}).$$

Then the spectrum $\sigma(\alpha, \pi)$ of SLP (2.2), (1.4) *and the spectrum $\sigma(P_{\alpha\pi} + Q_{\alpha\pi}, W_{\alpha\pi})$ of the matrix-pair $(P_{\alpha\pi} + Q_{\alpha\pi}, W_{\alpha\pi})$ are the same.*

Lemma 4.4. *Consider the separated BC* (1.4) *with $\alpha = 0$ and $\beta = \pi$. Define an $(n-1) \times (n-1)$ Jacobi matrix*

$$P_{0\pi} = \begin{bmatrix} p_1 + p_2 & -p_2 & & & \\ -p_2 & p_2 + p_3 & -p_3 & & \\ & \cdots & \cdots & \cdots & \\ & & -p_{n-2} & p_{n-2} + p_{n-1} & -p_{n-1} \\ & & & -p_{n-1} & p_{n-1} + p_n \end{bmatrix},$$

and diagonal matrices

$$Q_{0\pi} = \operatorname{diag}(q_1, q_{2,} \cdots, q_{n-1}), \qquad W_{0\pi} = \operatorname{diag}(w_1, w_2, \ldots, w_{n-1}).$$

Then the spectrum $\sigma(0, \pi)$ of SLP (2.2), (1.4) *and the spectrum $\sigma(P_{0\pi} + Q_{0\pi}, W_{0\pi})$ of the matrix-pair $(P_{0\pi} + Q_{0\pi}, W_{0\pi})$ are the same.*

Remark 4.1. We note that in Lemmas 4.1–4.4,

$$(P_{\alpha\beta} + Q_{\alpha\beta})_1 = P_{0\beta} + Q_{0\beta}, \qquad (P_{\alpha\beta} + Q_{\alpha\beta})^1 = P_{\alpha\pi} + Q_{\alpha\pi},$$

and

$$(P_{0\beta} + Q_{0\beta})^1 = P_{0\pi} + Q_{0\pi}, \qquad (P_{\alpha\pi} + Q_{\alpha\pi})_1 = P_{0\pi} + Q_{0\pi}.$$

Proof of Theorem 2.1. (a) For a given partition (1.6) of (a, b), define

$$w_i = \int_{a_i}^{b_i} w, \quad i = 0, 1, \ldots n, \quad \text{and} \quad W_{\alpha\beta} = \operatorname{diag}(w_0, w_1, \ldots, w_n).$$

By (1.9), $w_i > 0$, $i = 0, 1, \ldots, n$. Since $k = n + 1$, by Theorem 3.2, there exists a unique negative Jacobi matrix $M \in \mathbb{M}_{n+1}$ in the form of (3.1) such that

$$\sigma(M, W) = \{\lambda_i\colon i = 1, \ldots, n + 1\} \quad \text{and} \quad \sigma(M_1, W_1) = \{\mu_i\colon i = 1, \ldots, n\}.$$

Let

$$p_i = d_i, \quad i = 1, \ldots, n; \qquad q_i = c_{i+1} - p_i - p_{i+1}, \quad i = 1, \ldots, n-1,$$
$$q_0 = c_1 - p_1 - \cot\alpha, \qquad q_n = c_{n+1} - p_n - \cot\beta,$$

and define $P_{\alpha\beta}, Q_{\alpha\beta}, P_{0\beta}$, and $Q_{0\beta}$ by (4.1)–(4.4), respectively. Clearly, $p_i > 0$, $i = 1, \ldots, n$. It is easy to see that $M = P_{\alpha\beta} + Q_{\alpha\beta}$ and $M_1 = P_{0\beta} + Q_{0\beta}$. With the notation in (4.4) we also have $(W_{\alpha\beta})_1 = W_{0\beta}$. Therefore,

$$\sigma(P_{\alpha\beta} + Q_{\alpha\beta}, W_{\alpha\beta}) = \{\lambda_i\colon i = 1, \ldots, n+1\}$$

and

$$\sigma(P_{0\beta} + Q_{0\beta}, W_{0\beta}) = \{\mu_i\colon i = 1, \ldots, n\}.$$

By Lemmas 4.1 and 4.2 we have that for SLP (2.2), (1.4)

$$\sigma(\alpha, \beta) = \{\lambda_i\colon i = 1, \ldots, n+1\} \quad \text{and} \quad \sigma(0, \beta) = \{\mu_i\colon i = 1, \ldots, n\}.$$

We observe that the choice of p_i, $i = 1, \ldots, n$, and q_i, $i = 0, \ldots, n$, is unique and all $r, q \in L(a, b)$ by this choice form an equivalent family of SLPs. This completes the proof.

(b) The proof is similar using Corollary 3.1, and Lemmas 4.1, 4.3. We omit the details. □

The proof of Theorem 2.2 is similar to that of Theorem 2.1 using Theorem 3.2, Corollary 3.1, and Lemmas 4.2–4.4. We omit the details.

To prove Theorems 2.3 and 2.4 we use, in addition to Theorems 3.3 and 3.4 for the inverse cyclic Jacobi matrix problems, the following results from [6, Theorems 2.2 and 2.3] on the equivalence between Sturm–Liouville problems of Atkinson type and matrix eigenvalue problems.

Lemma 4.5. *Consider the real coupled BC* (1.5) *with* $k_{12} \neq 0$. *Define an* $(n+1) \times (n+1)$ *cyclic Jacobi matrix*

$$P_I = \begin{bmatrix} p_1 - k_{11}/k_{12} & -p_1 & & & & 1/k_{12} \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-1} & p_{n-1} + p_n & -p_n \\ 1/k_{12} & & & & -p_n & p_n - k_{22}/k_{12} \end{bmatrix} \tag{4.5}$$

and diagonal matrices

$$Q_I = \operatorname{diag}(q_0, q_1, \ldots, q_{n-1}, q_n), \qquad W_I = \operatorname{diag}(w_0, w_1, \ldots, w_{n-1}, w_n). \tag{4.6}$$

Then the spectrum $\sigma(K)$ *of SLP* (2.2), (1.5) *and the spectrum* $\sigma(P_I + Q_I, W_I)$ *of the matrix-pair* $(P_I + Q_I, W_I)$ *are the same.*

Lemma 4.6. *Consider the real coupled BC* (1.5) *with* $k_{12} = 0$. *Define an* $n \times n$ *cyclic Jacobi matrix*

$$P_\Theta = \begin{bmatrix} -k_{11}k_{21} + p_1 + k_{11}^2 p_n & -p_1 & & & & -k_{11}\, p_n \\ -p_1 & p_1 + p_2 & -p_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & & -p_{n-2} & p_{n-2} + p_{n-1} & -p_{n-1} \\ -k_{11}\, p_n & & & & -p_{n-1} & p_{n-1} + p_n \end{bmatrix}$$

and diagonal matrices

$$Q_\Theta = \operatorname{diag}\bigl(q_0 + k_{11}^2 q_n, q_1, \ldots, q_{n-1}\bigr), \qquad W_\Theta = \operatorname{diag}\bigl(w_0 + k_{11}^2 w_n, w_1, \ldots, w_{n-1}\bigr).$$

Then the spectrum $\sigma(K)$ *of SLP* (2.2), (1.5) *and the spectrum* $\sigma(P_\Theta + Q_\Theta, W_\Theta)$ *of the matrix-pair* $(P_\Theta + Q_\Theta, W_\Theta)$ *are the same.*

Proof of Theorem 2.3. (a) For a given partition (1.6) of (a, b), define

$$w_i = \int_{a_i}^{b_i} w, \quad i = 0, 1, \ldots n, \quad \text{and} \quad W_I = \operatorname{diag}(w_0, w_1, \ldots, w_n).$$

By (1.9), $w_i > 0$, $i = 0, 1, \ldots, n$. Since $k = n+1$, by Theorem 3.4, there exists a negative cyclic Jacobi matrix $N \in \mathbb{M}_{n+1}$ in the form of (3.3) such that

$$\sigma(N, W) = \{\lambda_i\colon i = 1, \ldots, n+1\} \quad \text{and} \quad \sigma(N_1, W_1) = \{\mu_i\colon i = 1, \ldots, n\}.$$

Let $p_i = -d_i$, $i = 1, \dots, n$; then $p_i > 0$, $i = 1, \dots, n$. Let $k_{12} = 1/d_{n+1}$, then $k_{12} < 0$. For $\beta \in (0, \pi)$ choose $K \in SL_2(\mathbb{R})$ such that $\cot\beta = k_{22}/k_{12}$ $k_{12} = 1/d_{n+1}$. Thus K defines a couple BC (1.5). Let

$$q_i = c_{i+1} - p_i - p_{i+1}, \quad i = 1, \dots, n-1,$$

$$q_0 = c_1 - p_1 + k_{11}/k_{12}, \qquad q_n = c_{n+1} - p_n + k_{22}/k_{12}.$$

Define $P_I, Q_I, P_{0\beta}$, and $Q_{0\beta}$ by (4.5), (4.6), (4.3), and (4.4), respectively. It is easy to see that $N = P_I + Q_I$ and $N_1 = P_{0\beta} + Q_{0\beta}$. With the notation in (4.4) we also have $(W_I)_1 = W_{0\beta}$. Therefore,

$$\sigma(P_I + Q_I, W_I) = \{\lambda_i\colon\ i = 1, \dots, n+1\}$$

and

$$\sigma(P_{0\beta} + Q_{0\beta}, W_{0\beta}) = \{\mu_i\colon\ i = 1, \dots, n\}.$$

By Lemmas 4.5 and 4.2 we have that for SLP (2.2), (1.5)

$$\sigma(K) = \{\lambda_i\colon\ i = 1, \dots, n+1\} \quad \text{and} \quad \sigma(0, \beta) = \{\mu_i\colon\ i = 1, \dots, n\}.$$

This completes the proof.

(b) The proof is similar using Corollary 3.3(a), and Lemmas 4.6, 4.3. We omit the details. □

The proof of Theorem 2.4 is similar to that of Theorem 2.1 using Theorem 3.4, Corollary 3.1, and Lemmas 4.6 and 4.4. We only need to note that the condition $k_{11} > 0$ is needed to guarantee that $p_n > 0$ in P_Θ. We omit the details.

5. Comments

SLPs of Atkinson type are clearly a very special subclass of all regular self-adjoint SLPs. However Volkmer [10] has shown, using the Radon–Nikodym theorem, that problems of Atkinson type include those studied by Feller [3] and Krein [8,9] in connection with their work on frequencies of vibrating strings and diffusion operators.

Most of the literature on inverse problems is restricted to the case when both the leading coefficient p and the weight function w are identically 1 on the whole interval (a, b) and the BCs are separated. From (1.7)–(1.9) it is clear that the fact that $1/p, q, w$ are identically zero on certain subintervals of (a, b) plays a fundamental role for all our theorems in Section 2.

One of the celebrated papers on inverse SLPs is the paper of Borg [2] in which he showed that, when p and w are identically 1, two given spectra for SLPs consisting of Eq. (1.2) and two preassigned separated BCs (1.4) determine the potential q uniquely.

Below we comment on some differences between our results on inverse SLPs of Atkinson type and the classical inverse Sturm–Liouville theory using Borg's theorem as an illustration.

1. In [2] there is an *a priori* assumption that the two given sets of infinite numbers are spectra. Theorems 2.1–2.4 are for two arbitrarily given finite sets of real numbers as long as they are strictly interlacing.

2. Borg's theorem guarantees that the Sturm–Liouville equation is uniquely determined by the two preassigned spectra, while Theorems 2.1 and 2.2 guarantee the existence of an equivalent family which contains an infinite number of such equations.

3. Borg's theorem requires that the two spectra are from two prescribed BCs whereas for Theorems 2.1 and 2.2 one BC is arbitrarily given, the other is related to the given one, and there is an arbitrarily chosen weight function w. The equivalent family of Sturm–Liouville equations is then determined by them.

4. Theorems 2.3 and 2.4 are for real coupled BCs. Not much seems to be known for the classical inverse SLPs with coupled BCs.

References

[1] F.V. Atkinson, Discrete and Continuous Boundary Value Problems, Academic Press, New York, London, 1964.
[2] G. Borg, Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe, Acta Math. 78 (1946) 1–96.
[3] W. Feller, The general diffusion operator and positivity preserving semi-groups in one dimension, Ann. of Math. 60 (1954) 417–436.
[4] W.E. Ferguson, The construction of Jacobi and periodic Jacobi matrices with prescribed spectrum, Math. Comp. 35 (1980) 1203–1220.
[5] H. Hochstadt, On some inverse problems in matrix theory, Arch. Math. 18 (1967) 201–207.
[6] Q. Kong, H. Volkmer, A. Zettl, Matrix representations of Sturm–Liouville problems with finite spectrum, Results Math. 54 (2009) 103–116.
[7] Q. Kong, H. Wu, A. Zettl, Sturm–Liouville problems with finite spectrum, J. Math. Anal. Appl. 263 (2001) 748–762.
[8] M.G. Krein, Determination of the density of a symmetrical inhomogeneous string from its spectrum of frequencies, Dokl. Akad. Nauk SSSR 76 (1951) 345–348.
[9] M.G. Krein, On inverse problems for an inhomogeneous string, Dokl. Akad. Nauk SSSR 82 (1952) 669–672.
[10] H. Volkmer, Eigenvalue problems of Atkinson, Feller and Krein and their mutual relationship, Electron. J. Differential Equations 48 (2005) 15–24 (electronic).
[11] S.F. Xu, An Introduction to Inverse Algebraic Eigenvalue Problems, Peking University Press, 1998.
[12] A. Zettl, Sturm–Liouville Theory, Math. Surveys Monogr., vol. 121, Amer. Math. Soc., 2005.